

A Bayesian Game without ϵ -equilibria

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Abstract

We present a three player Bayesian game for which there is no ϵ -equilibria in Borel measurable strategies for small enough positive ϵ , however there are non-measurable equilibria.

Key words: Bayesian games, non-amenable semi-group action, equilibrium existence

1 Introduction

Game theory is about strategies and the expected payoffs resulting from these strategies. The fundamental concept of game theory is that of an equilibrium, strategies for each player such that no player prefers to switch to another strategy, given that the strategies of the other players remain fixed. An ϵ -equilibrium for any $\epsilon \geq 0$ is defined in the same way – no player prefers by more than ϵ to switch to another strategy. An equilibrium is a 0-equilibrium.

What do we mean by a player preferring some strategy over another by at least ϵ ? Implied by a preference is an evaluation in an ordered field, usually the real numbers. Conventionally we assume that there is a topological probability space on which the game takes place, that each player is assigned a Borel sigma field corresponding to its knowledge of the space, and that that the players choose strategies that are measurable with respect to their respective Borel sigma fields. Given fixed measurable strategies of the other players, the evaluation of a strategy for a player is global, an integration of a function over the whole probability space. When the players attain an ϵ -equilibrium by this criteria, a *global* or *Harsanyi* ϵ -equilibrium is obtained.

On the other hand, a player could orient itself locally to some minimal sets in its sigma algebra, known as *information sets* (should they exist), and maximise its payoff with regard to these sets and its knowledge of them (which we presume is consistent with its knowledge of the whole probability space through regular conditional distributions). When each player at each such minimal set cannot obtain more than ϵ with the payoff evaluated locally at this set, a Bayesian ϵ -equilibrium is obtained.

Bayesian games are ancient, most card games being good examples. J. Harsanyi (1967) introduced a global theoretical perspective to Bayesian games, the origin of the term Harsanyi equilibrium. P. Milgrom and R. Weber (1985) asked implicitly the question whether Bayesian games always have measurable equilibria after proving existence for a special class of Bayesian games and analysing a game which did not belong to that class. R. Simon (2003) demonstrated an example of a three player Bayesian game for which there is no Borel measurable equilibrium however there are many non-measurable equilibria. This example was made possible by a structure of knowledge for which always the players have common knowledge of a countable set of measure zero. These sets of measure zero give no orientation for compiling the

many local equilibria toward Borel measurable strategies. Indeed, elementary applications of ergodic theory showed that Borel measurable equilibria could not exist.

A significant advance was performed by Z. Hellman (2014). He showed that there is a two-player Bayesian game with Bayesian equilibria but no Bayesian ϵ -equilibrium that is also Borel measurable for small enough positive ϵ . This discovery was advanced further by Z. Hellman and J. Levy (2016), in which it was demonstrated that a broad class of knowledge structures support games for which the same holds. This paper serves well as a general source to the structure, problems, and history of Bayesian games, especially in their relation to countable equivalence relations that are amenable.

It is required of a Bayesian ϵ -equilibrium that throughout the space each player cannot gain locally more than ϵ through an alternative strategy. However in a Harsanyi ϵ -equilibrium there could be a player who, according to the strategies defining the ϵ equilibrium, could improve its payoff by as much as $B > 0$ at a set of measure no more than ϵ/B . A Borel measurable Bayesian ϵ -equilibrium is a Harsanyi ϵ -equilibrium, but not vice versa. Indeed it is not difficult to show that for every $\epsilon > 0$ there are Harsanyi ϵ -equilibria to the Hellman game cited above.

One can perceive sets of very small measure where a player can act foolishly as a kind of firewall, absorbing the problems of the measurability requirement. Amenable structures tend to allow for such firewalls; for example with the closely related topic of Borel colouring; see Kechris, et al (1999). Therefore we would not have expected to find a game example lacking Harsanyi ϵ equilibria (yet possessing Bayesian equilibria) without utilizing a non-amenable structure.

In this paper we demonstrate that there is a Bayesian game played on a topological probability space Ω for which there are no Harsanyi ϵ -equilibria for all $\epsilon \leq \frac{1}{1000}$, and yet there are non-measurable Bayesian equilibria that employ pure strategies almost everywhere (pure meaning all weight given to one action). Because Ω is a Cantor set and the concern is the existence of approximate equilibria, moving from Borel to Lebesgue measurability does not alter the result.

As long as a game has an Harsanyi ϵ -equilibrium for every positive ϵ there is an equilibrium payoff, namely a cluster point of payoffs corresponding to the ϵ -equilibria as ϵ goes to 0. By this interpretation of an equilibrium payoff,

ours is a Bayesian game that has equilibria, but no equilibrium payoff.

With our example, there is no proper subset of the probability space for which the players have common knowledge, hence the arguments used are different from that of previous Bayesian games that lack Harsanyi equilibria but have Bayesian equilibria (which do utilize countable equivalence relations). Nevertheless we use a non-amenable semi-group action that contains some similarities to the structure of the Hellman example.

The rest of this paper is organised as follows. In the next section we define the game, followed in the third section by a proof that this game has no Harsanyi $\frac{1}{1000}$ -equilibrium. In the fourth section we show that it does have non-measurable Bayesian equilibria. The concluding fifth section is a presentation of open problems.

2 The Game:

Let G^+ be the free semi-group generated by the non-negative powers of the two generators T_1 and T_2 , with e the identity included in G^+ . Let X be the space $\{0, 1\}^{G^+}$, with an $x \in X$ a collection of the form $(x^e, x^{T_1}, x^{T_2}, x^{T_1T_2}, x^{T_2T_1}, x^{T_1^2}, x^{T_2^2}, \dots)$ with $x^U \in \{0, 1\}$ for every $U \in G^+$. For both $i = 1, 2$ define $T_i : X \rightarrow X$ to be the shift: $T_i(x)^V = x^{T_i V}$ for all $V \in G^+$, and for every $U, V \in G^+$ define $UV : X \rightarrow X$ by $UV(x)^V = x^{UV}$. This defines a right action of G^+ on X , meaning that $UV(x) = V \circ U(x)$. For every $x \in X$ define $G^+(x)$ to be the countable set $\{U(x) \mid U \in G^+\}$. Very important to the structure of X is that for every $x, y \in X$ there are two $z_1, z_2 \in X$ such that $T_1(z_i) = x$ and $T_2(z_i) = y$ for both $i = 1, 2$. We call the points *the twins determined* by x and y . We place the canonical probability distribution m on X which gives $\frac{1}{2}$ to each 0 or 1 placed in each position of G^+ and independently, so that a cylinder defined by n positions is given the probability 2^{-n} . With this probability distribution, we see that all $U \in G^+$ are measure preserving actions, meaning that $m(U^{-1}A) = m(A)$ for all Borel subsets A .

Of special importance, the independence probability assumption on each position implies that the distribution m on X can be reconstructed from the measure preserving property of $T_1 : X \rightarrow X$ and $T_2 : X \rightarrow X$, its product distribution m^2 , combined with the equal probability given to both twins.

Let D be the set $D := \{r, g\}$, r for red r and g for green. The probability

space on which the game is played will be $\Omega := D \times X$. We define the topology on Ω to be that induced by the clopen (closed and open) sets defined by the set D and the cylinder sets of X , so that Ω is a Cantor set. We define the canonical probability distribution μ on Ω so that for each choice of $d \in D$ and 0 or 1 in n distinct positions the probability for this cylinder set will be $\frac{1}{2} 2^{-n}$. For example, μ gives the set $\{(r, x) \mid x^e = 0, x^{T_1} = 1\}$ the probability $\frac{1}{8}$. The measure μ is the *common prior* for the game, meaning the Borel probability measure by which the game is defined.

There are three players, labelled G_0, R_1, R_2 . The information sets of each player are defined as follows. For each $x \in X$ Player G_0 considers (g, x) and (r, x) possible, and with equal $\frac{1}{2}$ probability, and these two points constitute its information set. For each $i = 1, 2$ and each x Player R_i consider (r, x) and $\{g\} \times T_i^{-1}(x)$ possible, with the point (r, x) and the set $\{g\} \times T_i^{-1}(x)$ of equal $\frac{1}{2}$ probability, and this pairing of a point with the corresponding Cantor set is its information set. Notice that this belief by the player R_i is consistent with the probability distribution on Ω , as the measure preserving property of the T_i implies that $m(T_i^{-1}(A)) = m(A)$ for all Borel subsets A of X . If B is the information set of a player, it means that this player cannot distinguish between any two points of this set and therefore has to conduct the same behaviour throughout the set. The Borel sigma algebra defining the knowledge of a player is that induced by these information sets, meaning the collection of all Borel sets B such that every information set of this player is either inside of B or disjoint from B .

All players have only two actions. The red players R_1 and R_2 have the actions $\mathbf{a}_0, \mathbf{a}_1$ and the green player G_0 has the actions $\mathbf{b}_0, \mathbf{b}_1$.

For either player R_i the only payoff that matters is that obtained at those states labelled r , and for the player G_0 the same is true for those states labelled g . There are two equivalent approaches to be taken, illustrated for a player R_i . Either the payoff obtained at (r, x) , described below, is duplicated at all the other points in the same information set, namely the set $\{g\} \times T_i^{-1}(x)$, or the payoffs obtained at (r, x) , described below, is multiplied by 2 and at all other points in the same information set the payoff is 0. Though the latter interpretation may be better suited to some theoretical approaches, as it employs the probability distribution μ on Ω , we will assume throughout the former equivalent interpretation (and for Player G_0 as well). This will allow us to focus on the set X and its probability distribution m .

The \mathbf{a}_0 and \mathbf{a}_1 pertain to actions of Player G_0 at both (g, x) and (r, x) . If $x^e = 0$ then the payoff matrices for the players R_i at the states (r, x) are

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} R_1 \\ \mathbf{b}_0 \\ \mathbf{b}_1 \end{array} & \begin{array}{cc} 300 & 0 \\ 0 & 100 \end{array} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} R_2 \\ \mathbf{b}_0 \\ \mathbf{b}_1 \end{array} & \begin{array}{cc} 100 & 0 \\ 0 & 300 \end{array} \end{array} .$$

If $x^e = 1$ then the payoff matrices at (r, x) are reversed:

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} R_1 \\ \mathbf{b}_0 \\ \mathbf{b}_1 \end{array} & \begin{array}{cc} 100 & 0 \\ 0 & 300 \end{array} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} R_2 \\ \mathbf{b}_0 \\ \mathbf{b}_1 \end{array} & \begin{array}{cc} 300 & 0 \\ 0 & 100 \end{array} \end{array} .$$

More complex are the payoffs of the player G_0 at a state labelled g . The matrix is three dimensional, meaning that it is a $2 \times 2 \times 2$ matrix. We need only to describe a 2×2 matrix corresponding to each action of the G_0 player. The rows and columns stand for the actions of the R_1 and R_2 players, respectively. Those actions \mathbf{a}_0 and \mathbf{a}_1 are performed by the R_1 player at both (g, x) and (r, T_1x) and by the R_2 player at both (g, x) and (r, T_2x) . First we describe the payoff matrices if $x^e = 0$:

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} \mathbf{b}_0 \\ \mathbf{a}_0 \\ \mathbf{a}_1 \end{array} & \begin{array}{cc} 1000 & 0 \\ 0 & 2000 \end{array} \end{array} \quad \begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} \mathbf{b}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_1 \end{array} & \begin{array}{cc} 0 & 1000 \\ 2000 & 0 \end{array} \end{array}$$

On the other hand, if $x^e = 1$ then the structure of payoffs is reversed:

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} \mathbf{b}_0 \\ \mathbf{a}_0 \\ \mathbf{a}_1 \end{array} & \begin{array}{cc} 0 & 1000 \\ 2000 & 0 \end{array} \end{array} \quad \begin{array}{cc} & \begin{array}{cc} \mathbf{a}_0 & \mathbf{a}_1 \end{array} \\ \begin{array}{c} \mathbf{b}_1 \\ \mathbf{a}_0 \\ \mathbf{a}_1 \end{array} & \begin{array}{cc} 2000 & 0 \\ 0 & 1000 \end{array} \end{array}$$

A strategy of a player is a function from its collection of information sets to the probability distributions on its two actions (a one dimensional simplex). The strategy is Borel measurable if that function is measurable with respect to its Borel sigma algebra (which is defined canonically as above from its information sets).

Notice that however the G_0 player acts at some (g, x) , that action is copied at (r, x) (because the G_0 player cannot distinguish between these two points). However the R_i players respond at (r, x) , those actions are copied at the

sets $\{g\} \times T_i^{-1}(x)$ respectively (as the R_i player cannot distinguish between (r, x) and $\{g\} \times T_i^{-1}(x)$). The behaviour of a player at (g, x) or (r, x) will influence inductively the behaviour of all players at an uncountable subset leading upward through repetitive applications of the T_i^{-1} . However the behaviour of players that influences inductively a player's payoff at (g, x) or (r, x) lie entirely within the countable set $D \times G^+(x)$. With regard to this latter aspect of influence, our game shares similarity with those defined by countable equivalence relations.

3 No Harsanyi $\frac{1}{1000}$ -equilibria

Before we show that the game has no Harsanyi $\frac{1}{1000}$ -equilibrium, we focus in on the subset $\{g\} \times X$.

Let A_0 be the subset of X such that the probability that Player G_0 at $\{g\} \times A_0$ chooses \mathbf{b}_0 is at least $\frac{19}{20}$. Let A_1 be the corresponding subset of X such that the probability that Player G_0 chooses \mathbf{b}_1 is at least $\frac{19}{20}$. Let A_M be the subset $X \setminus (A_0 \cup A_1)$.

As a general rule, from the above payoff matrices and the assumption that players are following their interests (the interests of the R_i players at (r, x) being that of conveying the choice of the G_0 player at (g, x)), we would expect that if $T_1(x) \in A_i$ and $T_2(x) \in A_j$, and $x^e = k$ then $x \in A_{i+j+k}$ where $i+j+k$ is represented modulo two. We call this the *parity rule*, and say that this rule holds for a point (g, x) whenever these three containments are true. We say that the parity rule holds for any $x \in X$ when it holds for (g, x) .

If any player chooses both actions at some point with strictly more than $\frac{1}{20}$ we say that the player is *mixing* at that point (meaning in A_M when this player is G_0). If there is a player and a set A of measure at least $w > 0$ where that player prefers one strategy over another by at least $r > 0$ and either that player is mixing or choosing the non-preferred action, then that player can gain at least $\frac{rw}{20}$ by choosing a different strategy. Therefore in an ϵ equilibrium it follows that w is at most $\frac{20\epsilon}{r}$. This simple fact is the bridge between the equilibrium concept and the semi-group action on X .

With respect to an ϵ -equilibrium for sufficiently small enough ϵ , there are two aspects of the game very important to our following arguments, First, where the strategies in approximate equilibrium are not mixing, they tend to fall

into the parity rule and stay there. Second, mixing is strongly discouraged by the structure of the payoffs. Looking at the payoffs of the R_1 and R_2 players at (r, x) , it is not possible for G_0 at (g, x) and (r, x) to make both other players indifferent to their two different actions. And then if z_0 and z_1 are twins, namely $T_i(z_0) = T_i(z_1)$ for both $i = 1, 2$, if the R_i player is not mixing at $(r, T_i(z_j))$ for at least one of $i = 1, 2$, it is not possible for the G_0 player at (g, z_j) to be indifferent to its two actions at both $j = 0, 1$. This dynamic is formalised in the next lemma.

Lemma 1: For every $x \in X$, either one or the other corresponding Player R_1 or R_2 at (r, x) has an incentive of at least 80 to choose either \mathbf{a}_0 or \mathbf{a}_1 over the other action. Let $x, y \in X$ be any two points in X and let z_0 and z_1 be the two twins where $T_1(z_i) = x$ and $T_2(z_i) = y$ for $i = 0, 1$ and $z_0^e = 0$ and $z_1^e = 1$. If one of R_1 or R_2 is mixing at (r, x) or (r, y) , respectively, and the other is not, then Player G_0 at either (g, z_0) or at (g, z_1) has an incentive of at least 80 to choose either \mathbf{b}_0 or \mathbf{b}_1 over the other strategy.

Proof: Without loss of generality assume that $x^e = 0$ and that the Player G_0 at (g, x) chooses \mathbf{b}_0 with probability at least $\frac{1}{2}$. By choosing \mathbf{a}_1 the R_1 player would get no more than 50 and by choosing \mathbf{a}_0 the R_1 player would get at least 150. On the other hand, if the Player G_0 at (g, x) chooses \mathbf{b}_1 with probability at least $\frac{1}{2}$ then the R_2 player would get no more than 50 by choosing \mathbf{a}_0 and at least 150 by choosing \mathbf{a}_1 .

Next, due to symmetries, it suffices to consider the two cases of the R_2 player choosing \mathbf{a}_0 with probability no more than $\frac{1}{20}$ and the R_2 player choosing \mathbf{a}_1 with probability no more than $\frac{1}{20}$.

Let $w \leq \frac{1}{20}$ be the probability that the R_2 player chooses \mathbf{a}_0 . We break this case into two subcases, where Player R_1 chooses \mathbf{a}_0 with at least $\frac{3}{5}$ and where Player R_1 chooses \mathbf{a}_0 with at most $\frac{3}{5}$. If Player R_1 chooses \mathbf{a}_0 with at least $\frac{3}{5}$ then the G_0 player at (g, z_1) gets at least 570 for playing \mathbf{b}_0 and no more than $400(1 - w) + 2000w$ for playing \mathbf{b}_1 , which reaches a maximum of 480 at $w = \frac{1}{20}$. If Player R_1 chooses \mathbf{a}_1 with at least $\frac{2}{5}$ then the G_0 player at (g, z_0) gets at least 760 from choosing \mathbf{b}_0 and no more than $600(1 - w) + 2000w$ for playing \mathbf{b}_1 , which reaches a maximum of 670 at $w = \frac{1}{20}$.

Now let $w \leq \frac{1}{20}$ be the probability that the R_2 player chooses \mathbf{a}_1 . We break this case into two subcases, where Player R_1 chooses \mathbf{a}_0 with at least $\frac{3}{5}$ and where Player R_1 chooses \mathbf{a}_0 with at most $\frac{3}{5}$. If Player R_1 chooses \mathbf{a}_0 with at least $\frac{3}{5}$ then the G_0 player at (g, z_1) get at least 1140 from choosing \mathbf{b}_1 and

no more than 820 by choosing \mathbf{b}_0 . If Player R_1 chooses \mathbf{a}_1 with at least $\frac{2}{5}$ then the G_0 player at (g, z_0) gets at least 780 from choosing \mathbf{b}_1 and no more than $600(1-w) + 2000w$ from choosing \mathbf{b}_0 , which reaches a maximum of 670 at $w = \frac{1}{20}$. \square

The consequence of Lemma 1 is that the players are hardly ever mixing in an approximate equilibrium. That is formalized in the next lemma.

Lemma 2: In any $\frac{1}{1000}$ Borel measurable equilibrium of the game, the G_0 player mixes with probability less than $\frac{16}{10,000}$ and the parity rule holds for all but at most $\frac{4}{1000}$ of the space X .

Proof: Let B_1 be the subset of $z \in A_M$ such that the corresponding R_1 player at (r, T_1z) is mixing and let B_2 be the subset of $z \in A_M$ such that the corresponding R_2 Player at (r, T_2z) is mixing. Let $c = m(A_M)$, $a = m(B_1)$ and $b = m(B_2)$. As the T_1z and T_2z are distributed independently as variables of z , in an $\frac{1}{1000}$ equilibrium the following holds: $c \leq \frac{1}{4000} + ab + \frac{1}{2}(a+b)$, where the $\frac{1}{4000}$ refers to the maximum probability for the G_0 player to choose a strategy that is suboptimal by a quantity of at least 80, the ab refers to the probability that both R_i are mixing at both T_1z , and $\frac{a+b}{2}$ refers to the probability that Player G_0 's actions are within 80 of each other for one but not both of the twins z_0 and z_1 (where one or the other of R_1 at T_1z_i or R_2 at T_2z_i are mixing, but not both). But as the sets B_i are the sets T_i^{-1} applied to where R_i is mixing in $\{r\} \times X$ and the T_i are measure preserving, a is also the probability throughout $\{r\} \times X$ that R_1 is mixing and the same holds for b and $\{r\} \times X$. By Lemma 1, for any $z \in X$ the probability of both R_1 mixing at (r, z) and R_2 mixing at (r, z) cannot exceed $\frac{1}{4000}$ (from $\frac{20}{80 \cdot 1000} = \frac{1}{4000}$). From this we conclude that $a+b \leq c + \frac{1}{1000}$, since where (g, z) is mixing at most $\frac{1}{2000}$ of the points following in both (r, z) and (r, z) can be mixing, (with the other $\frac{1}{2000}$ referring to the possibility that G_0 is not mixing at (g, z) nevertheless one of the R_i players at (r, z) is mixing).

From $ab \leq \frac{1}{4}(a+b)^2$, and the above, we get the quadratic $0 \leq c^2 - \frac{999}{500}c + \frac{3,001}{1,000,000}$. After completing the square we get that $|c - \frac{999}{1000}| \geq \sqrt{.995}$. Since c cannot be greater than 1 we are left with $c < .999 - .9974 = .0016$. The probability that the parity rule is not followed for an $x \in X$ is no more than the probability of the G_0 player mixing at either (g, T_1x) or (g, T_2x) plus the probability that the R_1 player at (r, T_1x) , the R_2 player at (r, T_2x) or the G_0 player at (g, x) is not properly responding to the corresponding non-mixing behaviour. These probabilities sum to .00395. \square

Next we show it is impossible for there to exist a $\frac{1}{1000}$ equilibrium Borel measurable equilibrium, using the regularity of the measure.

Let \mathcal{C}_n be the set of cylinders of depth n , where the two cylinders defined by the values $x^e = 0$ and $x^e = 1$ have depth 0. With $2^{n+1} - 1$ words of length n or less the cardinality of \mathcal{C}_n is $2^{2^{n+1}-1}$ and $m(c) = 2^{-2^{n+1}+1}$ for all $c \in \mathcal{C}_n$. Recall the definition of A_0 and A_1 as the sets where either 0 or 1 is played by G_0 with probability at least $\frac{19}{20}$. For every $c \in \mathcal{C}_n$ and $i = 0, 1$ let $w_i(c)$ be the conditional probability of action \mathbf{b}_i at the cylinder c , in other words $m(A_i \cap c)/m(c)$. For every cylinder c define $\eta(c) := \min_{i=0,1} w_i(c)$ and let $r(c)$ be the conditional probability of belonging in the set where the parity rule does not hold.

In the next lemma, we show that the parity rule is a powerful force to equalize the probabilities for both actions \mathbf{b}_0 and \mathbf{b}_1 . This cannot be guaranteed for all cylinders, due to the small probability that the parity rule doesn't hold. But it does hold in general for most cylinders, regardless of the depth. Two free generators and the dual causation implicit in the parity rule force this equalization.

Lemma 3: In any $\frac{1}{1000}$ equilibrium of the game, the average $q_i = \frac{\sum_{c \in \mathcal{C}_i} \eta(c)}{|\mathcal{C}_i|}$ is at least $\frac{1}{3}$ for every i .

Proof: The proof is by induction. There are two elements in \mathcal{C}_0 and eight elements in \mathcal{C}_1 . Let c_0 and c_1 be the two elements of \mathcal{C}_0 . Both cylinders c_0 and c_1 are composed of four elements of \mathcal{C}_1 , created (by means of T_1 and T_2) from the combination of the two elements c_0, c_1 of \mathcal{C}_0 and the same two elements c_0, c_1 of \mathcal{C}_0 along with a value of $x^e = 0$ for all $x \in c_0$ and $x^e = 1$ for all $x \in c_1$. Let x_0 and x_1 be two points such that $x_0^e = 0$, $x_1^e = 1$, and $x_0^U = x_1^U$ for every other $U \neq e$. However membership in A_0 or A_1 is determined by $T_1 x_i$ and $T_2 x_i$, the parity rule requires opposite memberships for $i = 0, 1$. As the parity rule must hold in a set of size at least $1 - \frac{1}{250}$, it follows that in the whole space the probability given to both A_0 and A_1 must be approximately the same, more precisely these probabilities must be at least $\frac{124}{250}$ for both A_0 and A_1 . Now let c be either c_0 or c_1 . As c is created by either the e position being 0 or 1 and the four combinations of c_0 and c_1 in both direction T_1 and T_2 , whatever are the probabilities given for the two $w_i(c_0)$ and the two $w_i(c_1)$, the fact that $\frac{w_i(c_0) + w_i(c_1)}{2} \geq \frac{124}{250}$ for both $i = 0, 1$, implies that the conditional probability given to both A_0 and A_1 at c must be at least $2(\frac{124}{250})^2 - \frac{1}{125} \geq .48$.

We assume the claim is true for every $t \in \mathcal{C}_i$. Every $t \in \mathcal{C}_{i+1}$ is created through the combination of a pair $c, d \in \mathcal{C}_i$ with a determination of 0 or 1 (though this determination will play no role in the following argument). Let i_c be the action that is less frequent at c , and define i_d the same way. Let j be the action following from the parity rule determined by the value of t^e and the combination of i_c with i_d (however that is determined by the e position). If $r(c) = r(d) = r(t) = 0$ then the parity rule would give j exactly $\eta(c)(1 - \eta(d)) + \eta(d)(1 - \eta(c))$, as it would give the other action the greater quantity $(1 - \eta(c))(1 - \eta(d)) + \eta(c)\eta(d)$. Due to the influence of the quantities $r(c), r(d), r(t)$ we cannot say for sure that j is the action less taken at t . But we can say that $\eta(t) \geq -r(t) + \eta(c)(1 - \eta(d) - r(d)) + \eta(d)(1 - \eta(c) - r(c)) \geq \eta(c) + \eta(d) - 2\eta(c)\eta(d) - r(t) - \frac{r(c)+r(d)}{2}$. But with $\sum_{c \in \mathcal{C}_j} r(c) \leq \frac{1}{250}|\mathcal{C}_j|$ for all j it follows that $q_{i+1} \geq -\frac{1}{125} + \frac{1}{|\mathcal{C}_i|^2} \sum_{c,d \in \mathcal{C}_i} \eta(c) + \eta(d) - 2\eta(c)\eta(d) = -\frac{1}{125} + \frac{1}{|\mathcal{C}_i|} \sum_{c \in \mathcal{C}_i} \eta(c) + \frac{1}{|\mathcal{C}_i|} \sum_{d \in \mathcal{C}_i} \eta(d) + \frac{1}{|\mathcal{C}_i|^2} (\sum_{c \in \mathcal{C}_i} \eta(c)) (\sum_{d \in \mathcal{C}_i} \eta(d)) = -\frac{1}{125} + 2q_i - 2q_i^2$. By induction we conclude that $q_{i+1} \geq \frac{4}{9} - \frac{1}{125} > \frac{1}{3}$. \square

Theorem 1: There can be no Borel (μ) measurable $\frac{1}{1000}$ -equilibrium.

Proof: With $\eta(c)$ defined as in the proof of Lemma 3, for the mutually exclusive measurable sets A_0, A_1 of X it follows from the regularity of the measure μ that $\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} \frac{\sum_{c \in \mathcal{C}_n} \eta(c)}{|\mathcal{C}_n|} = 0$. But by Lemma 3 it never falls below $\frac{1}{3}$. \square

Notice that where the players are obeying the parity rule, even approximately so, the location where the payoff to Player G_0 is close to 2000 or to 1000 is determined by one or the other of the two other players, by Player R_1 in the half $\{x \in X \mid x^e = 0\}$ and by Player R_2 in the half $\{x \in X \mid x^e = 1\}$. Where the parity rule holds the Lesbegue measurability of the payoff of Player G_0 implies the Lesbegue measurability of the strategies of the R_i players and hence also the Lesbegue measurability of the strategy of the G_0 player in response. For the G_0 player to have an “equilibrium payoff” by some interpretation one must define that concept quite differently from the existence of Harsanyi ϵ -equilibria.

4 Bayesian equilibria:

In this section we will show that we can colour the space $X = \{0, 1\}^{G^+}$ modulo a null set N using only two colours: 1 and 0 or red and blue, respec-

tively, so that the parity rule is obeyed, and furthermore extend this to an equilibrium of the whole space Ω . Recall that the *parity rule* is a function $c : X \rightarrow X$ such that $c(x) = c(T_1(x)) + c(T_2(x)) + x^e$ (modulo 2). Notice that such a colouring on such a set defines a Bayesian equilibrium on the subset $\{g, r\} \times (X \setminus N)$. We show then how to extend these strategies to a Bayesian equilibrium on all of the space $\Omega = \{g, r\} \times X$.

Recall the definition of the twins determined by some $x, y \in X$. We say that a subset A of X is closed if for every pair x, y in A the twins determined by x and y are also in A . By the *closure* \overline{A} of a set $A \subseteq X$ we mean the smallest closed set containing A . We say that $A \subseteq X$ is *pyramidic* if $x \in A$ implies that $U(x) \in A$, where $U \in G^+$. The main example of pyramidic set is $G^+(x)$, where x is an arbitrary element of X . Notice that whenever a set is pyramidic that its closure is also pyramidic.

Define now the set

$$N = \{x \in X : U(x) = V(x), \text{ for some distinct } U, V \in G^+\}.$$

This set is a null set with respect to the product measure m on X . Indeed, for any given two distinct words $U, V \in G^+$, the equality implies an agreement on infinitely many coordinates, and there are only countably many words $U \in G^+$.

We are ready to prove the following lemma:

Lemma 4: Let $X_1 = X \setminus N$ and let c be the parity rule defined on X . Then there exists a colouring of X_1 using the two colours $\{0, 1\}$ which is consistent with the rule c .

Proof: We will proceed by transfinite induction, assuming the Axiom of Choice and thus Zorn's Lemma. Let x_0 be any element of X_1 and obtain the set $P_0 := G^+(x_0)$. We define now the colouring of P_0 of $G^+(x_0)$ as follows:

- (i) colour all the points $T_1 U(x_0)$ in red, where $U \in G^+(x_0)$ and U is the identity or begins on the right with T_1 ;
- (ii) colour all the points $T_2 V(x_0)$ in blue, where $V \in G^+(x_0)$ is the identity or begins on the right with T_1 ;
- (iii) colour the remaining points of the pyramid P_0 in the way that they satisfy the rule c .

After colouring all the points of P_0 , extend the colouring to all the points in the closure $\overline{P_0}$ of P_0 .

Next create a partial ordering on colourings of pyramidal and closed subsets of X that obey the parity rule, with one colouring greater than another if the subset is larger and their colourings agree on their common intersection (the smaller subset). Any tower of such colourings will define a colouring that obeys the parity rule. As Zorn's Lemma implies that there is a maximal element (as a tower defines its own least upper bound), it suffices to show that maximality implies that all of X_1 has been coloured. Let P be any pyramidal and closed subset of X_1 with a colouring that obeys the parity rule. Assume that P is not all of X_1 . Let x be the first member of X_1 that is not in P .

We say that x has a *hitting point in* P if $U(x) \in P$ for some $U \in G^+$ and whenever $U = VW$ and W is not the identity then $V(x) \notin P$.

Now we have the two cases:

Case 1) x has no hitting point with respect to P . Then we colour the closure $G^+(x)$ in the same way as the initial pyramidal set P_0 .

Case 2) x has a hitting point in P . Then colour the elements of $G^+(x)$ taking into account the colours of the hitting points. Notice that the closure of P implies that if Ux is not in P then one of $T_i Ux$ is also not in P , so that if $T_i Ux$ is a hitting point then $T_j Ux$ is not a hitting point, for $i \neq j$. This allows us to colour x arbitrarily and then move downward in a consistent way, with the colouring of $T_i Ux$ determined already only if $T_j Ux$ is a hitting point or had just been just coloured arbitrarily (for $j \neq i$).

And then we colour the closure of $G^+(x) \cup P$ according to the parity rule c for a larger set that is closed, pyramidal, and consistent with both the parity rule and the pre-existing colouring. \square

Theorem 2: There exists a Bayesian equilibrium on all of Ω .

Proof: Following on from the proof of Lemma 4, let $\overline{X_1}$ be the closure of X_1 and extend the colouring of X_1 to one on $\overline{X_1}$. For all three players G_0 , R_1 and R_2 define pure strategies on $D \times \overline{X_1}$ accordingly. With x any point in

$X \setminus X_1$, let Γ_x be the game defined on $D \times G^+(x)$ such that the strategies on $D \times (G^+(x) \cap \overline{X_1})$ are already fixed by the above colouring. As the game has only countably many positions, by Simon (2003) there is a Nash equilibrium defined on the game Γ_x . But notice that it defines an equilibrium when including those strategies on $D \times (G^+(x) \cap \overline{X_1})$ that are already fixed. Extend this equilibrium to an equilibrium on the set $D \times \overline{(G^+(x) \cap \overline{X_1})}$ through best reply responses (noticing that nothing done at a point y has any influence on any player at points Uy for any $U \neq e$ - just consider x, T_1x, T_2x and follow by induction). We can even assume that these best reply responses are pure strategies. As with the proof of Lemma 4, we define a partial ordering on pyramidal and closed subsets P of X and the equilibria defined on $D \times P$. In the same way, we show that an equilibrium can be defined on all of Ω , using critically that any extension of an equilibrium from a closed and pyramidal set P will not disturb the pre-existing equilibrium property on P . \square

There are some points in X for which any equilibrium requires a mixed strategy. Let x, y be the two points defined by $x^e = 0, y^e = 1, T_1(x) = y, T_2(x) = x, T_1(y) = x$ and $T_2(y) = x$. No matter how x is coloured, because $T_1(y) = T_2(y) = x$ and $y^e = 1, y$ must be coloured with 1. But then $T_1(x) = y, T_2(x) = x$ and $x^e = 0$ forces x to be coloured differently from itself.

5 Conclusion: open questions

Is there an example of an ergodic game (Simon 2003) that has no Harsanyi ϵ -equilibrium for some positive ϵ ? The examples of Simon (2003) and of Hellman (2014) were ergodic games, and ergodic games have Bayesian equilibria (Simon 2003). We believe the answer is yes and that it can be done through the action of a non-amenable group which defines the information structure of the players. With the example of this paper, there was a very strong mixing structure that kept the probability high for both actions at all cylinders. We believe that the weaker mixing structure from a group action would be sufficient to obtain the same result.

In the example of this paper, there are three players. Can the same result be accomplished with two players? We believe that the answer is yes. The structure of our example is similar to that of a free action of the free product $G = C_2 * C_3$ on $\{0, 1\}^G$. We believe it can be done through associating C_3

with the knowledge of one player and C_2 with the other player, (or with two other finite groups whose free product is non-amenable).

Lastly, what is the relationship between Bayesian equilibria and the Banach-Tarski paradox? Let G be a group acting in a measure preserving way on a probability space X and for every player i let G_i be a finite subgroup of G so that the information sets of Player i are the orbits of G_i and G is generated by the G_i . Is there a Bayesian game so defined such that every Bayesian equilibrium defines through its fibres a paradoxical decomposition of the space where the group elements demonstrating the paradox belong to G ? On this question we are agnostic.

6 References

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